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Old and New Results on Multicritical Points

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Thirty years after the Liu–Fisher paper on the *bicritical* and *tetracritical* points in quantum lattice gases, these multicritical points continue to appear in a variety of new physical contexts. This paper reviews some recent multicritical phase diagrams, which involve, e.g., high-T_c superconductivity and various magnetic phases which may (or may not) coexist with it. One recent example concerns the SO(5) theory, which combines the 3-component antiferromagnetic and the 2-component superconducting order parameters. There, the competition between the isotropic, biconical and decoupled fixed points yields bicritical or tetracritical points. Recalling old results on the subject, it is shown that the *decoupled* fixed point is stable, implying a *tetracritical* point, contrary to recent claims, which are critically discussed. Other examples, concerning, e.g., the superconducting versus charge and spin density wave phases are also discussed briefly. In all cases, extensions of old results can be used to correct new claims.

KEY WORDS: Multicritical points; bicritical point; tetracritical point; renormalization group; decoupled fixed point.

1. INTRODUCTION

In addition to celebrating Michael Fisher's 70'th birthday, this year we also celebrate thirty years to the famous Wilson–Fisher paper[(1)] on the ϵ-expansion. That paper appeared a few months before I arrived as a post-doc in Fisher's group at Cornell, and shaped much of my scientific activity in the following few years. The present paper is dedicated to Michael Fisher, in recognition of his many contributions to statistical physics, in gratitude for the many things which I learned from him in those years and in the 30 years that followed, and in appreciation for his personal guidance and friendship.

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The Wilson–Fisher paper started three decades of activity, in which the ϵ-expansion was used for many types of interactions, and for many types of order parameters. In this paper I concentrate on one special class of these studies, involving *bicritical* and *tetracritical* points, which arise when a varying anisotropy causes a crossover from the critical behavior of an isotropic n-component order parameter to those of order parameters with less components, and hence with lower symmetries.[2, 3] Fisher himself started the modern theoretical era in this field in his paper with Liu (also written thirty years ago),[4] which gave a detailed mean field analysis for the case of the supersolid. He then wrote many more papers on the subject.[5–10] For the purposes of the present discussion I would like to emphasize his papers with Kosterlitz and Nelson, on the bi- and tetracritical points in anisotropic antiferromagnetic systems.[6, 7]

Bi- and tetracritical points have been revisited quite often during the last twenty years, whenever new physical systems required such studies. Here I give a critical review of some recent discussions of such multicritical points, in the context of the materials which exhibit high temperature superconductivity. While parts of the recent literature require new studies of bi- and tetracritical points, it turns out that many of the "new" questions were already discussed in the seventies. The present paper aims to bridge between the two relevant communities, relate some of the "new" questions to some "old" answers, and illuminate some questions which still require further study.

2. HISTORICAL REVIEW

The first detailed mean field analysis of bicritical and tetracritical points was given by Liu and Fisher.[4] In that case, the competing order parameters involve the superfluid and the crystal, within a quantum lattice gas model. They found three basic scenarios: In the simplest case, the two ordered phases meet at a first order transition line, which ends at a *bicritical* point (where the two critical lines between these phases and the disordered high temperature phase also meet). At this point, both order parameters become critical simultaneously. Alternatively, the two ordered phases are separated by a *mixed* "supersolid" phase, bounded by two critical lines which meet the two disordering critical lines at a *tetracritical* point. The third scenario, which required special choices of the parameters, is a mixture of the first two: a "bubble" of a mixed phase exists near the tetracritical point, ending at some lower temperature, turning into a first order transition. Being based on mean field theory, all the expressions for the phase boundaries are analytic in the parameters (temperature and pressure), and the lines reach the multicritical point at finite angles with each other.

Bi- and tetracritical points were studied extensively in the context of the anisotropic antiferromagnet (AAFM) in an external uniform field.[11] In that case, one observes longitudinal ordering along the easy axis at low fields, with a first order spin-flop transition into a phase with transverse ordering. Kosterlitz, Nelson, and Fisher[6,7] (KNF) gave a detailed renormalization group (RG) analysis of this problem, with both a uniform and a staggered field, and found a rich variety of phase diagrams, involving both bi- and tetracritical points. Beginning with the two order parameter vectors $\mathbf{S}_1$ and $\mathbf{S}_2$, with n_1 and n_2 components, respectively (with $n_1 = 1$ and $n_2 = n-1$ for the AAFM problem), they wrote the Ginzburg–Landau–Wilson Hamiltonian

$$\mathscr{H} = -\int d^d x \,[\tfrac{1}{2}\,(r_1 \mathbf{S}_1^2 + r_2 \mathbf{S}_2^2 + (\nabla \mathbf{S}_1)^2 + (\nabla \mathbf{S}_2)^2) \\ + u\,|\mathbf{S}_1|^4 + v\,|\mathbf{S}_2|^4 + 2w\,|\mathbf{S}_1|^2\,|\mathbf{S}_2|^2], \qquad (1)$$

and studied the RG flow of u, v, and w on the critical surface, to first order in $\epsilon = 4-d$, where d is the dimensionality of space. When both order parameters are critical (i.e., at the multicritical point), there exist six fixed points in the $u-v-w$ space, and the critical behavior is determined by the stable fixed point, which is approached under the RG flow from some basin of attraction. At order ϵ, KNF drew a diagram which indicated which fixed point is stable for different values of n_1 and n_2. As either of these numbers increases, stability switched from the *isotropic Heisenberg fixed point* (IFP) (with $u^* = v^* = w^*$, hence with full rotational symmetry in the full $n = n_1 + n_2$-component order parameter space), via the *biconical fixed point* (BFP), (with non-zero u^*, v^*, and w^*, representing some lower symmetry) and then to the *decoupled fixed point* (DFP), at which $w = 0$ and each order parameter has its own critical behavior, similar to that on the corresponding critical line. As was already known from related studies,[12] the IFP is stable for $n < n_c = 4 - 2\epsilon + \mathcal{O}(\epsilon^2)$. Thus, at $d = 3$ and $n = 3$ one is close to the stability boundary between the IFP and the BFP. When the initial parameters are not in the basin of attraction of the stable fixed point, the system never has an infinite correlation length, and therefore the transition has been identified as having a *fluctuation driven first order*.[13,14] Quantitatively, one can calculate the details of this transition by following the RG flow until all the fluctuations are integrated over, and then treating the resulting free energy (which is unstable at quartic order, and thus requires the addition of higher order terms) using a mean field analysis.

The detailed type of the multicritical point (i.e., bi- or tetracritical) is determined by the combination $\Delta = uv - w^2$: this point is tetracritical when $\Delta > 0$, and bicritical when $\Delta \leqslant 0$. KNF thus concluded that one should expect a bicritical point for the stable IFP, and a tetracritical point for the stable BCP. The latter also follows for the stable DFP, when the two critical lines just cross each other. However, the latter is not relevant for the AAFM, with $n = 3$, and therefore has not been considered in detail.

The shape of the critical lines as they approach the multicritical point is determined by *scaling*. If the quadratic anisotropy has the form $g(n_2 \mathbf{S}_1^2 - n_1 \mathbf{S}_2^2)$, then the critical lines approach the multicritical point tangentially, as $|T_i(g) - T_c| \sim g^{1/\psi_i}$. For the critical disordering lines, $\psi_i = \phi_g$, where $\phi_g = \nu\lambda_g$ and $d - \lambda_g$ is the anomalous scaling dimension of g; under the RG iterations, $g(\ell) = e^{\lambda_g \ell} g(0)$, where e^ℓ is the length rescaling factor.[2,3,15] However, when the bicritical point is characterized by the IFP, then the detailed phase diagram below the bicritical point may depend on the initial value of the parameter Δ. Although Δ is *irrelevant* in the RG sense near the IFP, it has slow transients which decay as $\Delta \sim e^{\lambda_\Delta \ell}$, with $\lambda_\Delta < 0$. If initially $\Delta(0) > 0$, then after a finite number of iterations ℓ one may still have $\Delta(\ell) > 0$, resulting with two critical lines bounding a mixed phase, *as near a tetracritical point!* However, the difference between these two lines vanishes with $\Delta(\ell)$, and therefore the exponents ψ_i describing them contain a combination of ϕ_g and of $\phi_\Delta = \nu\lambda_\Delta$.[16] The two critical lines below T_c thus approach each other *faster* than those above T_c. If $\Delta(0)$ is already small, then one might mistakenly identify these two lines with a single first order line, and the tetracritical line with a bicritical one.

As stated, KNF found that to order ϵ, there is always *only one stable fixed point*. This fact was placed in a more general context by Brézin *et al.*,[17] who proved this statement for *any* quartic combination of the order parameter components. In related work, Wallace and Zia[18] showed that to order ϵ^3 (at least for $n > 0$) the RG flow is like that of a particle moving in a potential, with fixed points interchanging stability as they cross each other in the parameter space. Indeed, all the existing analyses of such flows (with the exception of $n = 0$, where one of the two stable fixed points cannot be reached for physical reasons[15]) always find at most one stable fixed point, even at higher orders in ϵ. Detailed examples concern the cubic case[12] and the more general *nm*-component order parameter case, where one can follow these interchanges between fixed point stabilities in detail.[15]

Unlike the stability analysis of most fixed points, which relies on calculations of the stability exponents λ_i within the ϵ expansion, or numerically, it was realized quite early that one can discuss the stability of the DFP quite generally, using *non-perturbative scaling arguments*.[15] At the

DFP, the coupling term $w\,|\mathbf{S}_1|^2\,|\mathbf{S}_2|^2$ scales like the product of two energy-like operators, having the dimensions $(1-\alpha_{n_i})/\nu_{n_i}$, where α_{n_i} and ν_{n_i} are the specific heat and correlation length exponents of each order parameter separately. Thus, the combined operator has the dimension $d-\lambda_D$, where

$$\lambda_D = \frac{1}{2}\left(\frac{\alpha_{n_1}}{\nu_{n_1}} + \frac{\alpha_{n_2}}{\nu_{n_2}}\right) \tag{2}$$

is the scaling exponent which determines the RG flow of the coefficient of this term, w, near the DFP.

Indeed, such arguments gave the RG basis for the Harris criterion for quenched random systems (where the parameter which measures the randomness in the coupling constants scales with $\lambda = \alpha/\nu$),[15] and led to the prediction of a tetracritical point for a quenched random alloy of systems with competing spin anisotropies.[19] For the two order parameter problem discussed by KNF, one concludes that in $d=3$ the boundary of stability between the BFP and the DFP occurs in fact at much lower values of n_1 and n_2 than those expected from the order-ϵ estimates.

3. HIGH TEMPERATURE SUPERCONDUCTORS AND THE STORY OF SO(5)

The cuprate-based materials exhibit very rich phase diagrams, and it is generally believed that a good theory should not only explain the high-temperature superconductivity, but also explain the other phases which exist near or simultaneously with the superconducting one. In this connection, it was emphasized already in 1988 that doping introduces quenched randomness, with a potential magnetic spin glass phase.[20] In fact, this spin glass phase exhibits interesting scaling of the equation of state,[21] with interesting crossover to a lower symmetry of the order parameter due to the magnetic field.[22] The concentration–temperature phase diagram presented in ref. 20, containing many of the interesting phases which arise in these exciting materials, was later reproduced by Michael Fisher,[23] to demonstrate possible deviations from the general Gibbs rules in quenched random systems (Note the improved graphics introduced by Fisher in this reproduction!)

In its simplest version, the phase diagram of these materials contains only antiferromagnetic (AFM) and d-wave superconducting (SC) order. In 1997, Zhang[24] constructed an SO(5) theory, which aimed to unify the 3-component AFM order parameter and the 2-component complex SC order parameter into a combined 5-component theory. As the concentration of electronic holes increases from half filling (on the copper ions), one expects a transition from the AFM phase into the SC phase. Zhang argued that this transition is first order, ending at a bicritical point. This point is also where the two critical lines, with the critical behavior of the 3-component AFM and the 2-component SC ordered phases, meet, ending up with the critical behavior of the higher symmetry SO(5) group.

In its simple classical version, this SO(5) model maps onto the model discussed in the previous section, with $\mathbf{S}_1$ and $\mathbf{S}_2$ representing the 3-component and 2-component AFM and SC order parameters, respectively. Indeed, following Zhang's paper there appeared several papers which repeated some of the RG analysis reviewed above,[25, 26] with similar results. In particular, these references followed the order-ϵ analysis of KNF, and concluded that for $d=3$, $n_1=3$ and $n_2=2$ one has a *tetracritical* point, governed by the *biconical* fixed point. However, since for $n=5$ the BFP and the IFP may be close to each other, it has been suggested that one might actually observe a bicritical point, with exponents dominated by the IFP. Measurement of such exponents was even presented as "an experimental measurement of the number 5 of the SO(5) theory"![27] However, even in such a scenario, ref. 26 incorrectly stated that the all the four phase boundary exponents ψ_i are the same, equal to ϕ_g (in contrast to ref. 16).

Following this background, Hu[28] used Monte Carlo (MC) simulations on an SO(5) rotator model, and concluded that the multicritical point which characterizes the simultaneous ordering of the SO(3) AFM 3-component and of the U(1) SC 2-component order parameters, $\mathbf{S}_1$ and $\mathbf{S}_2$, has the critical behavior of the *isotropic* 5-component rotator model. This seems to contradict the RG in $d=4-\epsilon$ dimensions, which states that (a) to a high order in ϵ, the isotropic SO(n) fixed point (IFP) is *unstable* for $n>n_c$, with $n_c<4$,[15] and (b) to order ϵ, this multicritical point is described by the anisotropic biconical fixed point.[25, 26, 7]

These MC results by Hu (as well as the statements in many of the SO(5) papers in the literature) suffer from several problems. First, one might question the relevance of this discussion to high-T_c superconductivity (where one should also include fluctuations in the *electromagnetic gauge field*.[13]) Second, these papers ignore the *quenched randomness*, which is intrinsic for all of the doped cuprates (even if some electronic properties may be viewed as dominated by extended wave functions). Here we ignore these two points, and concentrate on the third issue: as reviewed in the previous section, at $d=3$ the multicritical point *must* be tetracritical, being characterized by the *decoupled* fixed point (DFP). Returning to Eq. (2), we can now use the known negative values of α_2 and α_3 at $d=3$,[29] to find that $\lambda_D \cong -0.087<0$, and the DFP is stable, in contrast to the order-ϵ extrapolation to $\epsilon=1$.[25, 26, 7] Thus, asymptotically the free energy breaks into a sum of the two free energies, $\mathbf{S}_1$ and $\mathbf{S}_2$ exhibit the Heisenberg ($n=3$) and

XY ($n = 2$) critical exponents and the two critical lines cross each other at finite angles, with the crossover exponent $\phi = 1$.[30] The latter statement is only asymptotic; after a finite number of RG iterations one still has a finite $w(\ell)$, yielding corrections to the phase boundaries which approach the asymptotic lines tangentially. Accurate experiments in the *asymptotic* regime thus carry no information on the SO(5) theory. However, they may yield some information on the transient non-asymptotic behavior near the *initial* Hamiltonian.

Reference 28 used a *discrete* spin model, with $|\mathbf{S}_1|^2 + |\mathbf{S}_2|^2 = 1$. This is believed to be in the same universality class as a Ginzburg–Landau–Wilson (GLW) theory, with the quartic term $U(|\mathbf{S}_1|^2 + |\mathbf{S}_2|^2)^2$ (where initially $U \to \infty$).[31] Reference 28 then added a coupling $W\,|\mathbf{S}_1|^2|\mathbf{S}_2|^2$. Quantum fluctuations[32] and RG iterations[15] then also generate a term $V(|\mathbf{S}_1|^4 - |\mathbf{S}_2|^4)$. Clearly, $u = U + V$, $v = U - V$ and $w = U + W/2$. Again, there exist six fixed points in the $U - V - W$ parameter space, of which *only one* should be stable.[15, 17, 18] For a continuous transition, the above argument implies an RG flow away from the vicinity of the unstable IFP, at $V = W = 0$, to the DFP, where $2U + W = 0$. This flow may be slow, since the related exponents λ_I^V and λ_I^W are small: the asymptotic DFP behavior can be observed only if $WX^{\lambda_I^W}$ becomes comparable to U, which is large. Here, $X = \min(L, \xi)$, with L the sample size and $\xi \sim (T - T_c)^{-\nu}$ the correlation length (T_c is the temperature at the multicritical point). Therefore, one might need to go very close to the predicted tetracritical point, and to much larger samples, in order to observe the correct critical behavior. The simulations of ref. 28, which begin close to the ITP ($U \gg V, W$) and use relatively small L, apparently stay in the *transient* regime which exhibits the isotropic exponents. To observe the true asymptotic decoupled behavior, one should start with a more general model, allowing different interactions for $\mathbf{S}_1$ and for $\mathbf{S}_2$, relax the strong constraint $|\mathbf{S}_1|^2 + |\mathbf{S}_2|^2 = 1$, and use much larger X. The latter is also needed due to the small value of λ_D. These requirements may be impossible for realistic MC simulations. In fact, Hu recently generalized his MC simulations, and used *finite* values of U.[33] However, his initial parameters obeyed $W < 4U$, which may still be much too close to the IFP. In these additional simulations, Hu still finds a bicritical phase diagram, with critical exponents which seem close to those of the IFP, thus contradicting the theoretical asymptotic expectation of a tertacritical point associated with the DFP.

There are three possible ways to explain this discrepancy:

- The crossover due to the RG flow from the initial vicinity of the IFP to the asymptotic DFP could be too slow, requiring much larger values of X than practical in the simulation. As X increases, I would expect signals of approaching the DFP. An example of such a signal would be the appearance of a "bubble" of the mixed phase near the multicritical point. Since this bubble may be narrow (and short), it could easily be identified as a single first order transition line. At low temperatures, the bubble could close back into the first order line, e.g., due to higher order terms in the Ginzburg–Landau expansion (as happened, e.g., in the Liu–Fisher phase diagram).

- If the initial Hamiltonian were out of the basin of attraction of the DFP, then one should observe first order transitions from the disordered phase into the ordered phases.[26] Again, the discontinuity on these transitions may be too small for the available values of X.

- Finally, there could be *two* stable fixed points. As stated several times above, I find this scenario most unlikely. In particular, it is well established that the IFP is *unstable* for $n > 4$. However, if indeed this scenario turns out to be true, then this case would represent a mini-revolution in our thinking of RG flows in such systems. It would be nice to have generalizations of the Brézin *et al.* and of the Wallace and Zia arguments to all orders in ϵ, and specifically for $d = 3$.

4. OTHER EXAMPLES

In addition to the simple AFM ordering, there have been many recent scattering experiments which exhibit some kind of (static or dynamic) *incommensurate peaks*.[34] These peaks, which may correspond to density and/or spin density wave ordering, usually arise at doping concentrations above those of the AFM phase, and often coexist with the SC phase. The general theory discussed above can thus be transferred to this new competition. In the simplest case, $\mathbf{S}_1$ would represent the *spin density wave* (SDW) order parameter, and $\mathbf{S}_2$ would continue to represent the SC ordering. Indeed, Kivelson *et al.*[35] generated a variety of temperature-concentration phase diagrams, taking account of the fact that the concentration x is related to the chemical potential μ which appears in the Ginzburg–Landau Hamiltonian via a Legendre transform. At the moment, there exists no detailed RG analysis of this case, which should be a generalization of the Fisher–Nelson[5] treatment of the AAFM at fixed magnetization. Apart from taking note of the "old" literature, such an analysis should also be careful in counting the components of the SDW order parameter. For an incommensurate wave vector, this number could be significantly larger than three,[14] and the RG may not have a stable fixed point at all, implying a fluctuation driven first order transition.

One theoretical scenario for the SDW ordering concerns *stripes*, which involve *charge density waves*.[35] This leads to a three-fold competition, between SDW, CDW and SC.[36] Since the wave vector of the CDW is equal to twice that of the SDW, this yields terms which are linear in the CDW order parameter and bilinear in the SDW one, possibly leading to first order transitions into the CDW phase.[36] Again, both the CDW and the SDW can have a large number of components, turning the RG treatment (not yet done) complicated but interesting.

Finally, I mention another class of phase diagrams, involving superconductivity in the bismuthates.[37] These systems exhibit both CDW and SC ordering, and their temperature-concentration phase diagrams have drawn much interest even before the discovery of high temperature superconductivity.[38] It turns out that both types of order can follow from a negative-U Hubbard model, which can then be mapped onto an anisotropic Ising–Heisenberg spin model. A mean field analysis of this model[37] yields phase diagrams which are similar to those found by Fisher and Nelson,[5] with their magnetization replaced by the concentration. The resulting coexistence region was ignored in earlier analyses.[38] In addition, the quenched randomness generates effective *random fields*, which couple to the CDW order parameters and cause a breakdown of that phase into finite domains, as apparently observed experimentally. It would be interesting to search for similar effects in the cuprates. It would also be interesting to have a comprehensive study of the role played by quenched randomness in these interesting systems.

5. CONCLUSIONS

- New materials bring about new phase diagrams, with competing types of order and with a variety of multicritical points. Cuprates and bismuthates are good examples of such rich varieties.

- Many details of these phase diagrams are often available from the early days of the RG research. It would help to bridge between the SC and the RG communities.

- In the context of SO(5), it would help to have more accurate experiments, as well as more MC simulations, in regimes which might be better suited for reaching the asymptotic correct behavior. In parallel, it might be of interest to find ways to investigate the relative stability of the competing fixed points *at* $d = 3$.

- After 30 years of RG studies, there are still new problems which require new RG treatments. It is appropriate to celebrate Fisher's 70th birthday recalling his "old" contributions, which opened the way to much of this "new" activity.

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